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Asymptotic integration of $(1+\alpha)$-order fractional differential equations

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ABSTRACT

We establish the long-time asymptotic formula of solutions to the $(1+\alpha)$-order fractional differential equation ${}_0^i\mathcal{O}_t^{1+\alpha}x+a(t)x=0$, $t>0$, under some simple restrictions on the functional coefficient $a(t)$, where ${}_0^i\mathcal{O}_t^{1+\alpha}$ is one of the fractional differential operators ${}_0D_t^{\alpha}(x')$, $({}_0D_t^{\alpha}x)'={}_0D_t^{1+\alpha}x$ and ${}_0D_t^{\alpha}(tx'-x)$. Here, ${}_0D_t^{\alpha}$ designates the Riemann–Liouville derivative of order $\alpha\in(0,1)$. The asymptotic formula reads as $[b+O(1)]\cdot x_{small}+c\cdot x_{large}$ as $t\to+\infty$ for given $b,c\in\mathbb{R}$, where x_{small} and x_{large} represent the eventually small and eventually large solutions that generate the solution space of the fractional differential equation ${}_0^i\mathcal{O}_t^{1+\alpha}x=0$, $t>0$.

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1. Introduction

The present note continues our recent papers [1–3] devoted to the fractional calculus variants of several fundamental results from the asymptotic integration theory of ordinary differential equations.

Let us consider the fractional differential equation (FDE) of order $1+\alpha$, with $\alpha\in(0,1)$, below

$$ {}_0^i\mathcal{O}_t^{1+\alpha}x+a(t)x=0,\quad t>0, \tag{1}$$

where the functional coefficient $a:[0,+\infty)\to\mathbb{R}$ is assumed continuous. The differential operator ${}_0^i\mathcal{O}_t^{1+\alpha}$ is a fractional version of the second order operator $\frac{d^2}{dt^2}$, built by taking into account the decompositions

$$x''=(x')',\qquad tx''=(tx'-x)',\quad t>0,$$

in the ring of smooth functions over $(0,+\infty)$.

To declare the operator ${}_0^i\mathcal{O}_t^{1+\alpha}$, denote by $\mathcal{RL}^{\alpha}((0,+\infty),\mathbb{R})$ the real linear space of all the functions $f\in C((0,+\infty),\mathbb{R})$ with $\lim_{t\searrow 0}[t^{1-\alpha}f(t)]\in\mathbb{R}$. Recall now the Riemann–Liouville derivative of order α of the function $f\in\mathcal{RL}^{\alpha}((0,+\infty),\mathbb{R})$, namely

$$({}_0D_t^{\alpha}f)(t)=\frac{1}{\Gamma(1-\alpha)}\cdot\frac{d}{dt}\left[\int_0^t\frac{f(s)}{(t-s)^{\alpha}}ds\right],\quad t>0,$$

where Γ stands for Euler's function Gamma; cf. [4, p. 68]. If the function f is at least absolutely continuous (see [5, p. 35, Lemma 2.2]) then the derivative exists almost everywhere. Now, we introduce the quantities

$${}_0^1\mathcal{O}_t^{1+\alpha}={}_0D_t^{\alpha}\circ\frac{d}{dt},\qquad {}_0^2\mathcal{O}_t^{1+\alpha}=\frac{d}{dt}\circ{}_0D_t^{\alpha}$$

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and

$$ {}_0^3\mathcal{O}_t^{1+\alpha} = {}_0D_t^\alpha \circ \left(t \cdot \frac{\mathrm{d}}{\mathrm{d}t} - \mathrm{id}_{\mathcal{RL}^\alpha((0,+\infty),\mathbb{R})} \right). $$

The different factorisations [6] of a fractional differential operator might lead to some interesting models in mathematical physics. We can mention that the fractional differential equations [7,8,5] are playing an important role in fluid dynamics, traffic model with fractional derivative, measurement of viscoelastic material properties, modeling of viscoplasticity, control theory, economy, nuclear magnetic resonance, mechanics, optics, signal processing and so on. Basically, the fractional differential equations are used to investigate the dynamics of the complex systems, the models based on these derivatives have given superior results as those based on the classical derivatives; see [4, p. 305], [9–11].

Notice that the FDE

$$ {}_0^i\mathcal{O}_t^{1+\alpha} x = 0, \quad t > 0, \tag{2} $$

has a bidimensional solution space in $\mathcal{RL}^\alpha((0, +\infty), \mathbb{R})$ generated by the smooth functions 1 and t^α for $i = 1$, $t^{\alpha-1}$ and t^α for $i = 2$, and t and $t^{\alpha-1}$ for $i = 3$.

Regarding Eq. (1) as *perturbation* of (2), one can ask *how close* a solution x of (1) can get to the solution $b \cdot x_{\text{small}} + c \cdot x_{\text{large}}$ of (2), with $b, c \in \mathbb{R}$? Some simple restrictions on the functional coefficient $a(t)$ will be given next to ensure that an asymptotic formula for the general solution of each of the three FDEs exist similarly to the case of classical ordinary differential equations. In a loose manner, the formula reads as

$$ [b + O(1)] \cdot x_{\text{small}} + c \cdot x_{\text{large}} \quad \text{when } t \to +\infty. \tag{3} $$

Given the fact that the singular integral operators employed in our proofs resemble the integral operators from the two-point boundary value problems encountered in the theory of second order differential equations (see [12]) we think that the Landau symbol $O(1)$ in our formula cannot be replaced with its counterpart $o(1)$ in the majority of circumstances.

2. The case of ${}_0^1\mathcal{O}_t^{1+\alpha}$

To establish (3), we introduce an integral operator acting within a complete metric space and prove that it is *a contraction* with respect to the space metric. The existence of its fixed point will follow then from the Contraction Principle and the solution based on the fixed point will obey the asymptotic formula.

We start with a formal derivation of the integral operator. Given $x \in C([0, +\infty), \mathbb{R})$ such that $x' \in \mathcal{RL}^\alpha((0, +\infty), \mathbb{R})$, we integrate (1) over $[t, +\infty)$ to get

$$ \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{x'(s)}{(t-s)^\alpha} \mathrm{d}s = x_1 + \int_t^{+\infty} (ax)(s)\mathrm{d}s, \quad t > 0, $$

where $x_1 = \lim_{t\to+\infty} \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{x'(s)}{(t-s)^\alpha} \mathrm{d}s \in \mathbb{R}$.

Further,

$$ \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-s)^{1-\alpha}} \int_0^s \frac{x'(u)}{(s-u)^\alpha} \mathrm{d}u\mathrm{d}s = x_1 \cdot \frac{t^\alpha}{\alpha} + \int_0^t \frac{1}{(t-s)^{1-\alpha}} \int_s^{+\infty} (ax)(u)\mathrm{d}u\mathrm{d}s. $$

A Fubini–Tonelli argument (see [5, p. 29]) leads to

$$ \begin{aligned} \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-s)^{1-\alpha}} \int_0^s \frac{x'(u)}{(s-u)^\alpha} \mathrm{d}u\mathrm{d}s &= \frac{1}{\Gamma(1-\alpha)} \int_0^t x'(u) \int_u^t \frac{\mathrm{d}s}{(t-s)^{1-\alpha}(s-u)^\alpha} \mathrm{d}u \\ &= \frac{1}{\Gamma(1-\alpha)} \int_0^t x'(u) \int_0^1 \frac{\mathrm{d}v}{(1-v)^{1-\alpha} v^\alpha} \mathrm{d}s \\ &= \frac{B(\alpha, 1-\alpha)}{\Gamma(1-\alpha)} \int_0^t x'(u)\mathrm{d}u, \end{aligned} $$

where B is the Beta function; cf. [4, p. 6]. Since $B(q, r) = \frac{\Gamma(q)\Gamma(r)}{\Gamma(q+r)}$ and $\Gamma(1+q) = q\Gamma(q)$, with $q, r \in (0, 1)$, we obtain that

$$ x(t) = x_0 + \frac{x_1}{\Gamma(1+\alpha)} \cdot t^\alpha + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{1}{(t-s)^{1-\alpha}} \int_s^{+\infty} (ax)(\tau)\mathrm{d}\tau\mathrm{d}s, $$

with $x(0) = x_0 \in \mathbb{R}$.

Taking $b = x_0,\ c = \frac{x_1}{\Gamma(1+\alpha)}$, with $a^2 + b^2 > 0$, the integral operator reads as

$$ \mathcal{T}(x)(t) = b + ct^\alpha + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{1}{(t-s)^{1-\alpha}} \int_s^{+\infty} (ax)(\tau)\mathrm{d}\tau\mathrm{d}s, \quad t > 0. \tag{4} $$

Theorem 1. *Assume that there exists $T > 0$ such that $\int_T^{+\infty} s^{1+\alpha}|a(s)|\mathrm{d}s < +\infty$ and*

$$\frac{\max\{1, T^\alpha\}}{\Gamma(1+\alpha)}\left[\int_0^T |a(s)|\mathrm{d}s + \int_T^{+\infty} s^\alpha |a(s)|\mathrm{d}s\right] = k < 1.$$

Then the FDE (1) for $i = 1$ has a solution $x \in C([0, +\infty), \mathbb{R})$ with the asymptotic formula

$$x(t) = b + ct^\alpha + O(t^{\alpha-1}) = b + ct^\alpha + o(1) \quad \text{when } t \to +\infty. \tag{5}$$

In particular, $O(1)$ can be replaced with $o(1)$ in (3).

Proof. Let X be the set of all the functions $x \in C([0, +\infty), \mathbb{R})$ with $\sup_{t\geq T} \frac{|x(t)|}{t^\alpha} < +\infty$ and d the following metric

$$d(x_1, x_2) = \max\left\{\|x_1 - x_2\|_{L^\infty(0,T)}, \sup_{t\geq T} \frac{|x_1(t) - x_2(t)|}{t^\alpha}\right\}, \quad x_1, x_2 \in X.$$

Obviously, $\mathcal{M} = (X, d)$ is a complete metric space.

Notice that

$$\begin{aligned}\int_0^{+\infty} s^j |ax|(s)\mathrm{d}s &\leq \left[\int_0^T s^j |a(s)|\mathrm{d}s + \int_T^{+\infty} s^{j+\alpha}|a(s)|\mathrm{d}s\right] d(x, 0)\\ &= C(j)\cdot d(x, 0),\end{aligned}$$

where $j \in \{0, 1\}$, for every $x \in \mathcal{M}$.

Introduce the operator $\mathcal{T} : \mathcal{M} \to C([0, +\infty), \mathbb{R})$ with the formula (4). We have the estimates

$$\begin{aligned}|\mathcal{T}(x)(t)| &\leq |b| + |c|t^\alpha + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\mathrm{d}s}{(t-s)^{1-\alpha}} \cdot \int_0^{+\infty} |ax|(s)\mathrm{d}s\\ &\leq |b| + T^\alpha\left[|c| + \frac{C(0)}{\Gamma(1+\alpha)}\cdot d(x, 0)\right], \quad t \in [0, T],\end{aligned}$$

and

$$|\mathcal{T}(x)(t)| \leq t^\alpha\left[\frac{|a| + T^\alpha|b|}{T^\alpha} + \frac{C(0)}{\Gamma(1+\alpha)}\cdot d(x, 0)\right], \quad t \geq T,$$

which imply that $\mathcal{T}(x) \in \mathcal{M}$ and

$$d(\mathcal{T}(x), 0) \leq \max\left\{1, \frac{1}{T^\alpha}\right\}(|b| + T^\alpha|c|) + \max\{1, T^\alpha\}\frac{C(0)}{\Gamma(1+\alpha)}\cdot d(x, 0),$$

where $x \in \mathcal{M}$.

We also have

$$d(\mathcal{T}(x_1), \mathcal{T}(x_2)) \leq \frac{\max\{1, T^\alpha\}}{\Gamma(1+\alpha)}C(0)\cdot d(x_1, x_2), \quad x_1, x_2 \in \mathcal{M},$$

which means that $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ is a contraction of coefficient k.

Let $x_0 \in \mathcal{M}$ be its fixed point. Following verbatim the computations from [2, Eqs. (10), (16)], we have the estimates

$$\begin{aligned}\int_0^t \frac{1}{(t-s)^{1-\alpha}}\int_s^{+\infty} |ax_0|(\tau)\mathrm{d}\tau\mathrm{d}s &= \int_0^t |ax_0|(\tau)\frac{t^\alpha - (t-\tau)^\alpha}{\alpha}\mathrm{d}\tau + \frac{t^\alpha}{\alpha}\int_t^{+\infty} |ax_0|\mathrm{d}s\\ &\leq \frac{t^\alpha}{\alpha}\left[\int_0^t |ax_0|(\tau)\cdot\frac{\tau}{t}\mathrm{d}\tau + \frac{1}{t}\int_t^{+\infty} s|ax_0|(s)\mathrm{d}s\right]\\ &\leq \frac{2C(1)}{\alpha}d(x_0, 0)\cdot t^{\alpha-1} = O(t^{\alpha-1}) \quad \text{when } t \to +\infty.\end{aligned}$$

Finally,

$$x_0(t) = \mathcal{T}(x_0)(t) = b + ct^\alpha + O(t^{\alpha-1}) \quad \text{when } t \to +\infty.$$

The proof is complete. □

A particular case of (5) has been undertaken in [1], namely the case when $b = 1$, $c = 0$. We asked there if, similarly to the circumstances of ordinary differential equations [12, Section 7], the solution x_0 from Theorem 1 would have the (powerful) asymptotic behavior

$$x_0(t) = 1 + o(1) \quad \text{as } t \to +\infty, \qquad x' \in (L^1 \cap L^\infty)((0, +\infty), \mathbb{R}).$$

We also noticed that, most probably, to get such a result one must look for a *sign-changing* functional coefficient $a(t)$; see [1, Section 3].

In the remaining of the present section we shall discuss the issue of "$x' \in L^1$" and conclude that this can happen (eventually) in very restricted conditions.

Lemma 1. *Assume that* $a \in (C \cap L^\infty)([0,+\infty),\mathbb{R})$ *verifies the hypotheses from* [1]*: it has a unique zero* $t_0 > 0$, $\int_0^{+\infty} a(s)\mathrm{d}s = 0$, $\int_0^{+\infty} s|a(s)|\mathrm{d}s < +\infty$ *and* $B \in (L^1 \cap L^\infty)([0,+\infty),\mathbb{R})$, *where* $B(t) = t^\alpha \|a\|_{L^\infty(t,+\infty)}$ *for all* $t \ge 0$. *Then, introducing the quantity* $C(t) = \int_0^t \frac{a(s)}{(t-s)^{1-\alpha}}\mathrm{d}s,\ t \ge 0$, *we have*

$$\int_0^{+\infty} |C(t)|\mathrm{d}t + \sup_{t\ge 0}|C(t)| < +\infty. \tag{6}$$

If $B^* \in L^1([0,+\infty),\mathbb{R})$, *where* $B^*(t) = \sup_{s\ge t} B(s)$ *for all* $t \ge 0$, *then*

$$\int_0^{+\infty} C^*(t)\mathrm{d}t < +\infty, \qquad C^*(t) = \sup_{s\ge t}|C(s)|,\ t \ge 0. \tag{7}$$

If $\int_0^{+\infty} s^{1+\alpha}|a(s)|\mathrm{d}s + \int_0^{+\infty} \|B\|_{L^2(t,+\infty)}\mathrm{d}t < +\infty$ *then we also have*

$$\int_0^{+\infty} \|C\|_{L^2(u,+\infty)}\mathrm{d}u = \int_0^{+\infty} \left(\int_u^{+\infty} |C(t)|^2\mathrm{d}t\right)^{\frac{1}{2}}\mathrm{d}u < +\infty. \tag{8}$$

Proof. As in [1], for $t > 0$, the following estimates are valid

$$|C(2t)| \le \frac{B(t)}{\alpha} + \left|\int_0^t \frac{a(s)}{(2t-s)^{1-\alpha}}\mathrm{d}s\right| \tag{9}$$

and

$$\int_0^t \frac{a(s)}{(2t-s)^{1-\alpha}}\mathrm{d}s = t^{\alpha-1}\int_0^t a(s)\mathrm{d}s - (1-\alpha)\int_0^t \frac{1}{(2t-s)^{2-\alpha}}\int_0^s a(\tau)\mathrm{d}\tau\mathrm{d}s \tag{10}$$

$$= -t^{\alpha-1}\int_t^{+\infty} a(s)\mathrm{d}s + (1-\alpha)\int_0^t \frac{1}{(2t-s)^{2-\alpha}}\int_s^{+\infty} a(\tau)\mathrm{d}\tau\mathrm{d}s. \tag{11}$$

Since $B \in L^1 \cap L^\infty$, it is obvious that $B \in L^2$, so we shall focus on the second member from the right part of (9). By means of (10), we get

$$\left|t^{\alpha-1}\int_0^t a(s)\mathrm{d}s\right| + \left|\int_0^t \frac{1}{(2t-s)^{2-\alpha}}\int_0^s a(\tau)\mathrm{d}\tau\mathrm{d}s\right| \le t^\alpha\|a\|_{L^\infty} + \frac{1}{t^{2-\alpha}}\int_0^t (s\cdot\|a\|_{L^\infty})\mathrm{d}s = \frac{3}{2}\|a\|_{L^\infty}\cdot t^\alpha,$$

which leads to $C \in (L^1 \cap L^\infty)([0,T_0],\mathbb{R})$, where $T_0 = \max\{1,t_0\}$. Further, via (11),

$$\begin{aligned} D(t) &= \left|\int_0^t \frac{a(s)}{(2t-s)^{1-\alpha}}\mathrm{d}s\right| \\ &\le t^{\alpha-1}\int_t^{+\infty}|a(s)|\mathrm{d}s + t^{\alpha-2}\int_t^{+\infty} s|a(s)|\mathrm{d}s \\ &\le \frac{2}{t^{2-\alpha}}\int_t^{+\infty} s|a(s)|\mathrm{d}s, \quad t \ge T_0, \end{aligned}$$

and so $C \in (L^1 \cap L^\infty)([T_0,+\infty),\mathbb{R})$. The estimate (6) has been obtained. As a byproduct, $C \in L^2([0,+\infty),\mathbb{R})$.

To prove (7), introduce $D^*(t) = \sup_{s\ge t} D(s)$ for all $t \ge 0$. We rely on the estimates

$$D^*(t) \le \frac{3}{2}\|a\|_{L^\infty}\cdot t^\alpha, \quad t \in [0,T_0],$$

and

$$\begin{aligned} \int_{T_0}^{+\infty} D^*(t)\mathrm{d}t &\le 2\int_{T_0}^{+\infty}\frac{\mathrm{d}s}{s^{2-\alpha}}\cdot\int_{T_0}^{+\infty}\tau|a(\tau)|\mathrm{d}\tau \\ &= \frac{2T_0^{\alpha-1}}{(1-\alpha)}\int_{T_0}^{+\infty}\tau|a(\tau)|\mathrm{d}\tau, \end{aligned}$$

since the mapping $t \mapsto t^{\alpha-2}\int_t^{+\infty} s|a(s)|\mathrm{d}s$ is monotone non-increasing in $[T_0,+\infty)$.

For the third part, notice that

$$D(t) \leq \frac{2}{t^2}\int_t^{+\infty} s^{1+\alpha}|a(s)|\mathrm{d}s, \quad t \geq T_0,$$

and

$$\begin{aligned}\left(\int_{2u}^{+\infty}|C(2t)|^2\mathrm{d}t\right)^{\frac{1}{2}} &\leq \frac{1}{\alpha}\cdot\|B\|_{L^2(2u,+\infty)} + \left(\int_{2u}^{+\infty}\frac{\mathrm{d}t}{t^4}\right)^{\frac{1}{2}}\cdot 2\int_{2u}^{+\infty} s^{1+\alpha}|a(s)|\mathrm{d}s \\ &\leq \alpha^{-1}\|B\|_{L^2(2u,+\infty)} + \frac{u^{-\frac{3}{2}}}{\sqrt{6}}\int_0^{+\infty} s^{1+\alpha}|a(s)|\mathrm{d}s, \quad u \geq T_0.\end{aligned}$$

We have obtained that $\int_{T_0}^{+\infty}\left(\int_{2u}^{+\infty}|C(2t)|^2\mathrm{d}t\right)^{\frac{1}{2}}\mathrm{d}u < +\infty$.

Finally,

$$\begin{aligned}\int_0^{T_0}\left(\int_{2u}^{+\infty}|C(2t)|^2\mathrm{d}t\right)^{\frac{1}{2}}\mathrm{d}u &= \frac{1}{\sqrt{2}}\int_0^{T_0}\left(\int_{4u}^{+\infty}|C(v)|^2\mathrm{d}v\right)^{\frac{1}{2}}\mathrm{d}u \\ &\leq \frac{T_0}{\sqrt{2}}\|C\|_{L^2(0,+\infty)}.\end{aligned}$$

The proof is complete. □

Lemma 2. *Assume that the function C from Lemma 1 satisfies the restrictions (6)–(8) and either*

$$\|C\|_{L^\infty} + 2\|C^*\|_{L^1} = k_1 < 1$$

or

$$2\|C^*\|_{L^1} < 1, \quad \max\left\{\|C\|_{L^\infty} + \|C\|_{L^2}, \|C\|_{L^1} + \|E\|_{L^1}\right\} = k_2 < 1,$$

where $E(t) = \|C\|_{L^2(t,+\infty)}$ for all $t \geq 0$. Then there exists a function $y \in (C \cap L^1 \cap L^\infty)([0,+\infty),\mathbb{R})$ such that

$$y(t) = -C(t)\left(1 - \int_t^{+\infty} y(s)\mathrm{d}s\right) - \int_t^{+\infty}(Cy)(s)\mathrm{d}s, \quad t \geq 0. \tag{12}$$

Proof. Set the number $\gamma > 1$ such that

$$1 + 2\gamma\int_0^{+\infty} C^*(s)\mathrm{d}s < \gamma.$$

Introduce the set Y of all the functions $y \in C([0,+\infty),\mathbb{R})$ such that $|y(t)| \leq \gamma\cdot C^*(t)$, $t \geq 0$, and the metric d with the formula

$$d(y_1, y_2) = \max\{\|y_1 - y_2\|_{L^\infty(0,+\infty)}, \|y_1 - y_2\|_{L^1(0,+\infty)}\}, \quad y_1, y_2 \in Y.$$

Using the Dominated Convergence Theorem, we deduce that the metric space $\mathcal{N} = (Y, d)$ is complete.

Consider the integral operator $\mathcal{T} : \mathcal{N} \to C([0,+\infty),\mathbb{R})$ given by the right-hand member of (12). The following estimates

$$\begin{aligned}|\mathcal{T}(y)(t)| &\leq |C(t)|(1 + \|y\|_{L^1}) + C^*(t)\int_t^{+\infty}|y(s)|\mathrm{d}s \\ &\leq C^*(t)(1 + 2\|y\|_{L^1}) \leq C^*(t)\cdot\left(1 + 2\gamma\int_0^{+\infty} C^*(s)\mathrm{d}s\right) \\ &\leq \gamma\cdot C^*(t), \quad t \geq 0,\end{aligned}$$

show that $\mathcal{T} : \mathcal{N} \to \mathcal{N}$ is well-defined.

Now, we have

$$\begin{aligned}|\mathcal{T}(y_1)(t) - \mathcal{T}(y_2)(t)| &\leq C^*(t)\|y_1 - y_2\|_{L^1} + \int_0^{+\infty}|C(s)|\mathrm{d}s\cdot\|y_1 - y_2\|_{L^\infty} \\ &\leq (\|C\|_{L^\infty} + \|C\|_{L^1})\cdot d(y_1, y_2),\end{aligned}$$

by noticing that $C^*(0) = \|C\|_{L^\infty(0,+\infty)}$, and also

$$\int_t^{+\infty} |\mathcal{T}(y_1)(s) - \mathcal{T}(y_2)(s)|\mathrm{d}s \le \int_t^{+\infty} (|C(s)| \cdot \|y_1 - y_2\|_{L^1})\mathrm{d}s + \int_t^{+\infty} C^*(s) \int_s^{+\infty} |y_1(\tau) - y_2(\tau)|\mathrm{d}\tau\mathrm{d}s$$
$$\le 2\int_0^{+\infty} C^*(s)\mathrm{d}s \cdot d(y_1, y_2), \quad t \ge 0,$$

which lead to

$$\begin{aligned} d(\mathcal{T}(y_1), \mathcal{T}(y_2)) &\le \max\left\{\|C\|_{L^\infty} + \|C\|_{L^1}, 2\|C^*\|_{L^1}\right\} \cdot d(y_1, y_2) \\ &\le k_1 d(y_1, y_2), \end{aligned}$$

where $y_1, y_2 \in \mathcal{N}$.

Notice that we have not employed (8). To do so, let us use different estimates, namely

$$|\mathcal{T}(y_1)(t) - \mathcal{T}(y_2)(t)| \le |C(t)| \cdot \|y_1 - y_2\|_{L^1} + \left[\int_t^{+\infty} |C(s)|^2\mathrm{d}s\right]^{\frac{1}{2}} \cdot \left[\int_t^{+\infty} |y_1(s) - y_2(s)|^2\mathrm{d}s\right]^{\frac{1}{2}}$$

and

$$\begin{aligned} \int_t^{+\infty} |y_1(s) - y_2(s)|^2\mathrm{d}s &\le \sup_{\tau\ge 0} |y_1(\tau) - y_2(\tau)| \cdot \int_t^{+\infty} |y_1(s) - y_2(s)|\mathrm{d}s \\ &\le [d(y_1, y_2)]^2, \quad t \ge 0. \end{aligned}$$

They imply

$$\begin{aligned} |\mathcal{T}(y_1)(t) - \mathcal{T}(y_2)(t)| &\le \left[|C(t)| + \left(\int_t^{+\infty} |C(s)|^2\mathrm{d}s\right)^{\frac{1}{2}}\right] \cdot d(y_1, y_2) \\ &\le (\|C\|_{L^\infty} + \|C\|_{L^2}) d(y_1, y_2) \end{aligned}$$

and

$$\int_t^{+\infty} |\mathcal{T}(y_1)(s) - \mathcal{T}(y_2)(s)|\mathrm{d}s \le \left[\|C\|_{L^1} + \int_0^{+\infty} \left(\int_t^{+\infty} |C(s)|^2\mathrm{d}s\right)^{\frac{1}{2}} \mathrm{d}t\right] d(y_1, y_2),$$

thus leading to

$$\begin{aligned} d(\mathcal{T}(y_1), \mathcal{T}(y_2)) &\le \max\left\{\|C\|_{L^\infty} + \|C\|_{L^2}, \|C\|_{L^1} + \|E\|_{L^1}\right\} \cdot d(y_1, y_2) \\ &\le k_2 d(y_1, y_2), \end{aligned}$$

where $y_1, y_2 \in \mathcal{N}$.

The operator $\mathcal{T} : \mathcal{N} \to \mathcal{N}$ being a contraction, its fixed point y_0 is the solution of (12) we are looking for. The proof is complete. □

Proposition 1. *Let $y \in C([0, +\infty), \mathbb{R})$ be the solution of (12) from Lemma 2. If $y(0) = 0$ then the function $x \in C^1([0, +\infty), \mathbb{R})$ with the formula $x(t) = 1 - \int_t^{+\infty} y(s)\mathrm{d}s$ for all $t \ge 0$ is a solution of the FDE (1) for $i = 1$ which satisfies the restrictions*

$$x(t) = 1 + o(1) \quad \text{as } t \to +\infty, \qquad x' \in (L^1 \cap L^\infty)([0, +\infty), \mathbb{R}).$$

Proof. Following [1], the function x verifies the identity

$$\begin{aligned} y(t) &= -\frac{1}{\Gamma(\alpha)} \int_0^t \frac{a(s)x(s)}{(t-s)^{1-\alpha}}\mathrm{d}s = -\frac{1}{\Gamma(\alpha)} \int_0^t \frac{a(s)}{(t-s)^{1-\alpha}}\mathrm{d}s + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{a(s)}{(t-s)^{1-\alpha}} \left(\int_s^t + \int_t^{+\infty}\right) y(\tau)\mathrm{d}\tau\mathrm{d}s \\ &= -C(t) + \int_0^t y(\tau)C(\tau)\mathrm{d}\tau + C(t)\int_t^{+\infty} y(\tau)\mathrm{d}\tau \\ &= -C(t)\left(1 - \int_t^{+\infty} y(s)\mathrm{d}s\right) + \int_0^t (Cy)(s)\mathrm{d}s, \quad t \ge 0. \end{aligned} \tag{13}$$

We have rescaled C as $C(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{a(s)}{(t-s)^{1-\alpha}}\mathrm{d}s,\ t \ge 0$.

Let $t = 0$ in (12). Then, $0 = y(0) = -\int_0^{+\infty}(Cy)(s)\mathrm{d}s$. This means that we can recast the integral expression from (13) as

$$y(t) = -C(t)\left(1 - \int_t^{+\infty} y(s)\mathrm{d}s\right) - \int_t^{+\infty}(Cy)(s)\mathrm{d}s, \quad t \geq 0,$$

which is exactly (12).

The proof is complete. □

To give some insight to the (still unsettled) issue of "$x' \in L^1$", notice that the condition $y(0) = 0$ from Proposition 1 reads as

$$\int_0^{+\infty} x'(s)\int_0^s \frac{a(\tau)}{(s-\tau)^{1-\alpha}}\mathrm{d}\tau\mathrm{d}s = 0,$$

which is really difficult to handle. A further intricacy is provided by the fact that, given $a \in C([0, +\infty), \mathbb{R})$, *the quantity* $F(t) = \int_0^t \frac{|a(s)|}{(t-s)^{1-\alpha}}\mathrm{d}s$, $t \geq 0$, *does not belong to* $L^1([0, +\infty), \mathbb{R})$. This follows from

$$\int_T^t F(2s)\mathrm{d}s \geq \int_T^t \int_{\frac{T}{2}}^{2s} \frac{|a(\tau)|}{(2s-\tau)^{1-\alpha}}\mathrm{d}\tau\mathrm{d}s \geq \int_T^t \frac{\mathrm{d}s}{\left(2s-\frac{T}{2}\right)^{1-\alpha}} \cdot \int_{\frac{T}{2}}^{2T} |a(\tau)|\mathrm{d}\tau \to +\infty \quad \text{when } t \to +\infty,$$

where $T > 0$ is chosen large enough for a to be non-trivial in $\left[\frac{T}{2}, 2T\right]$.

3. The case of ${}_0^3\mathcal{O}_t^{1+\alpha}$

Introduce the relations

$$y(t) = tx'(t) - x(t), \qquad x(t) = ct - t\int_t^{+\infty} \frac{y(\tau)}{\tau^2}\mathrm{d}\tau, \quad t > 0, \tag{14}$$

with $c \neq 0$ and $y \in \mathcal{RL}^\alpha((0, +\infty), \mathbb{R})$; see [3].

As before,

$$\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{y(s)}{(t-s)^\alpha}\mathrm{d}s = x_1 + \int_t^{+\infty}(ax)(s)\mathrm{d}s, \quad t > 0,$$

where $x_1 = \lim_{t\to+\infty} \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{y(s)}{(t-s)^\alpha}\mathrm{d}s \in \mathbb{R}$, and

$$\begin{aligned}\int_0^t y(s)\mathrm{d}s &= \frac{x_1 t^\alpha}{\Gamma(1+\alpha)} + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{1}{(t-s)^{1-\alpha}}\int_s^{+\infty}(ax)(\tau)\mathrm{d}\tau\mathrm{d}s \\ &= \frac{x_1 t^\alpha}{\Gamma(1+\alpha)} + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{1}{(t-s)^{1-\alpha}}\left(\int_0^{+\infty} - \int_0^s\right)(ax)(\tau)\mathrm{d}\tau\mathrm{d}s \\ &= \frac{t^\alpha}{\Gamma(1+\alpha)}\left[x_1 + \int_0^{+\infty}(ax)(\tau)\mathrm{d}\tau\right] - \frac{1}{\Gamma(\alpha)}\int_0^t\int_0^s \frac{(ax)(u)}{(s-u)^{1-\alpha}}\mathrm{d}u\mathrm{d}s;\end{aligned}$$

see [5, p. 32, Eq. (2.13)].

By differentiation, we get

$$y(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}\left[x_1 + \int_0^{+\infty}(ax)(\tau)\mathrm{d}\tau\right] - \frac{1}{\Gamma(\alpha)}\int_0^t \frac{(ax)(s)}{(t-s)^{1-\alpha}}\mathrm{d}s,$$

where $t > 0$.

Taking $b = -\frac{x_1}{(2-\alpha)\Gamma(\alpha)}$ and recalling (14), our integral operator reads as

$$\begin{aligned}\mathcal{T}(y)(t) &= t^{\alpha-1}\left[b + \frac{c}{\Gamma(\alpha)}\int_0^{+\infty} sa(s)\mathrm{d}s\right] - \frac{c}{\Gamma(\alpha)}\int_0^t \frac{sa(s)}{(t-s)^{1-\alpha}}\mathrm{d}s \\ &\quad - \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^{+\infty}\tau a(\tau)\int_\tau^{+\infty}\frac{y(u)}{u^2}\mathrm{d}u\mathrm{d}\tau + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\tau a(\tau)}{(t-\tau)^{1-\alpha}}\int_\tau^{+\infty}\frac{y(u)}{u^2}\mathrm{d}u\mathrm{d}\tau, \quad t > 0.\end{aligned}$$

Theorem 2. *Assume that* $\int_0^{+\infty} t|a(t)|\mathrm{d}t + \sup_{t>0} t^{1-\alpha}\int_0^t \frac{s|a(s)|}{(t-s)^{1-\alpha}}\mathrm{d}s < +\infty$ *and*

$$\frac{1}{\Gamma(\alpha)}\left(\int_0^{+\infty}\frac{|a(s)|}{s^{1-\alpha}}\mathrm{d}s + \chi\right) = k_3 < 1,$$

where $\chi = \sup_{t>0} t^{1-\alpha}\int_0^t \frac{|a(s)|}{(t-s)^{1-\alpha}s^{1-\alpha}}\mathrm{d}s$. *Then the FDE (1) for* $i = 3$ *has a solution* $x \in C^1((0,+\infty),\mathbb{R})$ *with the asymptotic formula*

$$x(t) = [b + O(1)]t^{\alpha-1} + ct = ct + O(t^{\alpha-1}) \quad \text{when } t \to +\infty. \tag{15}$$

Proof. Let us start by giving a simple example of χ. If the functional coefficient $a \in (C \cap L^1)([0,+\infty),\mathbb{R})$ verifies the restriction

$$|a(t)| \le \frac{A}{t^\alpha}, \quad t > 0,$$

then

$$\begin{aligned}
t^{1-\alpha}\int_0^{2t} \frac{|a(s)|}{(2t-s)^{1-\alpha}s^{1-\alpha}}\mathrm{d}s &= t^{1-\alpha}\left(\int_0^t + \int_t^{2t}\right)\frac{|a(s)|}{(2t-s)^{1-\alpha}s^{1-\alpha}}\mathrm{d}s \\
&\le t^{1-\alpha}\int_0^t \frac{|a(s)|}{t^{1-\alpha}s^{1-\alpha}}\mathrm{d}s + t^{1-\alpha}\int_t^{2t}\frac{A}{(2t-s)^{1-\alpha}s}\mathrm{d}s \\
&\le \left(\int_0^1 \frac{|a(s)|}{s^{1-\alpha}}\mathrm{d}s + \int_1^{1+t}\frac{|a(s)|}{s^{1-\alpha}}\mathrm{d}s\right) + A\int_{\frac{1}{2}}^1 \frac{\mathrm{d}v}{(1-v)^{1-\alpha}v} \\
&\le \left(\int_0^1 \frac{\mathrm{d}s}{s^{1-\alpha}}\cdot \|a\|_{L^\infty(0,1)} + \int_1^{+\infty}|a(s)|\mathrm{d}s\right) + 2A\int_{\frac{1}{2}}^1 \frac{\mathrm{d}v}{(1-v)^{1-\alpha}} \\
&\le \frac{1}{\alpha}\|a\|_{L^\infty(0,1)} + \|a\|_{L^1(1,+\infty)} + A\frac{2^{1-\alpha}}{\alpha} < +\infty, \quad t > 0.
\end{aligned}$$

Notice also that $\int_0^t \frac{|a(s)|}{(t-s)^{1-\alpha}}\mathrm{d}s \le t^{1-\alpha}\int_0^t \frac{|a(s)|}{(t-s)^{1-\alpha}s^{1-\alpha}}\mathrm{d}s \le \chi$ and

$$t^{1-\alpha}\int_0^t \frac{s|a(s)|}{(t-s)^{1-\alpha}}\mathrm{d}s = t^{1-\alpha}\int_0^t \frac{s^{2-\alpha}|a(s)|}{(t-s)^{1-\alpha}s^{1-\alpha}}\mathrm{d}s, \quad t > 0,$$

which leads to the "χ" of the mapping $t \mapsto t^{2-\alpha}a(t)$ in $[0,+\infty)$.

Introduce now the set Z of all the functions $y \in C((0,+\infty),\mathbb{R})$ such that $\sup_{t>0} t^{1-\alpha}|y(t)| < +\infty$ and the metric

$$d(y_1,y_2) = \sup_{t>0} t^{1-\alpha}|y_1(t) - y_2(t)|, \quad y_1, y_2 \in Z.$$

Observe also that

$$\begin{aligned}
\sup_{t>0} t^{2-\alpha}\int_t^{+\infty}\frac{|y_1(u)-y_2(u)|}{u^2}\mathrm{d}u &\le \frac{1}{2-\alpha}\cdot\sup_{t>0} t^{1-\alpha}|y_1(t)-y_2(t)| \\
&\le d(y_1,y_2).
\end{aligned} \tag{16}$$

The metric space $\mathcal{P} = (Z,d)$ is complete. Given $y \in \mathcal{P}$, we have the estimates

$$\begin{aligned}
t^{1-\alpha}|\mathcal{T}(y)(t)| \le |b| &+ \frac{|c|}{\Gamma(\alpha)}\int_0^{+\infty} s|a(s)|\mathrm{d}s + \frac{|c|}{\Gamma(\alpha)}\cdot\sup_{t>0} t^{1-\alpha}\int_0^t \frac{s|a(s)|}{(t-s)^{1-\alpha}}\mathrm{d}s \\
&+ \frac{1}{\Gamma(\alpha)}\int_0^{+\infty}\frac{|a(s)|}{s^{1-\alpha}}\mathrm{d}s\cdot\sup_{s>0} s^{2-\alpha}\int_s^{+\infty}\frac{|y(u)|}{u^2}\mathrm{d}u \\
&+ \frac{1}{\Gamma(\alpha)}\cdot\sup_{t>0} t^{1-\alpha}\int_0^t \frac{|a(\tau)|}{(t-\tau)^{1-\alpha}\tau^{1-\alpha}}\mathrm{d}\tau\cdot\sup_{\tau>0}\tau^{2-\alpha}\int_\tau^{+\infty}\frac{|y(u)|}{u^2}\mathrm{d}u, \quad t > 0,
\end{aligned}$$

which imply that $\mathcal{T}(\mathcal{P}) \subseteq \mathcal{P}$.

Further, we have

$$\begin{aligned}
t^{1-\alpha}|\mathcal{T}(y_1)(t) - \mathcal{T}(y_2)(t)| &\le \left[\frac{1}{\Gamma(\alpha)}\int_0^{+\infty}\frac{|a(s)|}{s^{1-\alpha}}\mathrm{d}s + \frac{1}{\Gamma(\alpha)}\sup_{t>0} t^{1-\alpha}\int_0^t \frac{|a(\tau)|}{(t-\tau)^{1-\alpha}\tau^{1-\alpha}}\mathrm{d}\tau\right] d(y_1,y_2) \\
&= \frac{1}{\Gamma(\alpha)}\left(\int_0^{+\infty}\frac{|a(s)|}{s^{1-\alpha}}\mathrm{d}s + \chi\right) d(y_1,y_2), \quad t > 0,
\end{aligned}$$

by means of (16), where $y_1, y_2 \in \mathcal{P}$.

The operator $\mathcal{T} : \mathcal{P} \to \mathcal{P}$ being a contraction of coefficient k_3, it has a fixed point y_0. Thus, since $y_0(t) = O(t^{\alpha-1})$ for large values of t, we conclude the validity of the asymptotic expansion (15) for the solution x given by (14). Notice also that

$$\lim_{t\searrow 0} t^{1-\alpha} y_0(t) = \lim_{t\searrow 0} t^{1-\alpha} \mathcal{T}(y_0)(t) = a + \frac{b}{\Gamma(\alpha)} \int_0^{+\infty} sa(s)\mathrm{d}s.$$

The proof is complete. $\square$

4. The case of ${}_0^2\mathcal{O}_t^{1+\alpha}$

The asymptotic formula (3) has been already discussed in [2], however, it is worthy to be recalled for reasons of completeness.

Theorem 3 *([2, Theorem 1]). Assume that there exists $T > 0$ such that*

$$\frac{\max\{1, T\}}{\Gamma(1+\alpha)} \left(\int_0^T \frac{|a(s)|}{s^{1-\alpha}} \mathrm{d}s + \int_T^{+\infty} s^\alpha |a(s)| \mathrm{d}s \right) = k_4 < 1$$

and $\int_T^{+\infty} s^{1+\alpha}|a(s)|\mathrm{d}s < +\infty$. Then, given $b, c \in \mathbb{R}$, with $b^2 + c^2 > 0$, the FDE (1) for $i = 2$ has a solution $x \in C((0, +\infty), \mathbb{R})$ with the asymptotic formula

$$x(t) = [b + O(1)]t^{\alpha-1} + ct^\alpha = ct^\alpha + O(t^{\alpha-1}) \quad \text{when } t \to +\infty.$$

The formula of the integral operator reads in this case as

$$\mathcal{T}(x)(t) = bt^{\alpha-1} + ct^\alpha + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{1}{(t-s)^{1-\alpha}} \int_s^{+\infty} (ax)(\tau)\mathrm{d}\tau \mathrm{d}s, \quad t > 0,$$

and its fixed point x_0 satisfies also the conditions

$$\lim_{t\searrow 0} t^{1-\alpha} x_0(t) = b, \qquad \lim_{t\to+\infty} ({}_0D_t^\alpha x_0)(t) = \Gamma(1+\alpha)c.$$

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